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CALCULATION OF PLANCK'S CONSTANT C_2

By J. H. Dellinger, Assistant Physicist

I. CALCULATION OF C_2 FROM ANY TWO OBSERVATIONS

The basis of a number of methods for measuring high temperatures is Planck's equation for radiation from a black body,

$$J = \frac{c_1}{\lambda^5\left(e^{\frac{c_2}{\lambda\theta}} - 1\right)} \qquad (1)$$

As this has been used very generally, much of the available data on properties of matter at high temperatures depend upon the physical constant c_2. This constant is important also in purely theoretical physics, in theories bearing on the constitution of the atom. The determination of the value of this constant is therefore of considerable interest.

Most of the determinations of c_2 have been from observations of the intensity of radiation J as a function of wave length λ, at constant temperature θ. Such observations were plotted in a curve with J as ordinates and λ as abscissas. c_2 was obtained by the method of equal ordinates; that is, by calculation from two wave lengths λ_1 and λ_2 corresponding to any pair of equal ordinates on opposite sides of the maximum (Fig. 1). The most convenient solution for this method was shown in a former paper[1] to be

$$c_2 = \frac{\lambda_1\lambda_2\theta}{\lambda_2 - \lambda_1}\left[5 \log \frac{\lambda_2}{\lambda_1} - e^{-\frac{c_2}{\lambda_2\theta}}\right] \qquad (2)$$

It is the purpose of the present paper to present a more general method of calculation which has several advantages.

[1] Buckingham and Dellinger, this Bulletin, 7, p. 393; 1911.

Since equation (1) contains two constants, it is obvious that two observations of J for any values of λ and θ suffice to find c_1 and c_2. Only c_2 is of interest, as c_1 merely gives the scale of ordinates, while c_2 determines the shape of the curve. It turns out that c_2 is calculable very simply from any two observations of J, λ, and θ. In particular, when the two observations are for the same

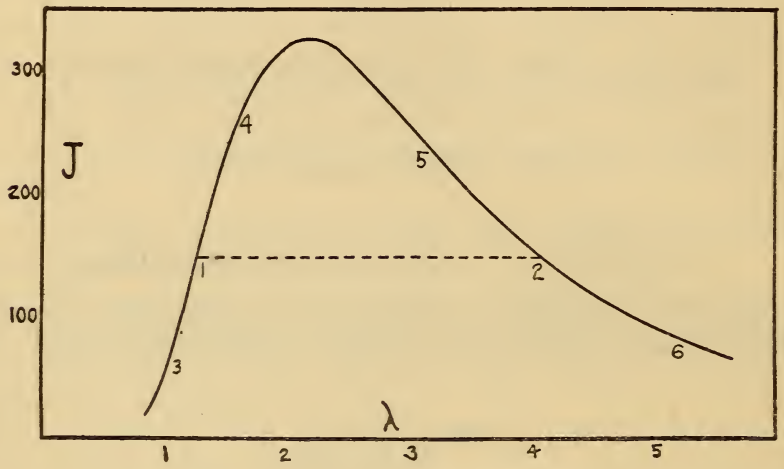

FIG. 1.—*Planck energy curve for 1350° K*

temperature, c_2 is obtained from any two points on the J, λ curve by merely adding a term to the equation for equal ordinates, thus:

$$c_2 = \frac{\lambda_1\lambda_2\theta}{\lambda_2 - \lambda_1}\left[\log\frac{J_2}{J_1} + 5\log\frac{\lambda_2}{\lambda_1} - e^{-\frac{c_2}{\lambda_2\theta}}\right] \qquad (3)$$

An explicit solution for c_2 has not been obtained, and would probably be complicated and useless if it could be found. A very valuable general expression is found as follows. Writing equation (1) in the form,

$$J_1 = \frac{c_1}{\lambda_1^5 e^{\frac{c_2}{\lambda_1\theta_1}}\left(1 - e^{-\frac{c_2}{\lambda_1\theta_1}}\right)}$$

it follows that

$$e^{\frac{c_2}{\lambda_1\theta_1}} = \frac{c_1}{J_1\lambda_1^5\left(1 - e^{-\frac{c_2}{\lambda_1\theta_1}}\right)}$$

$$\therefore \frac{c_2}{\lambda_1\theta_1} = \log c_1 - \log J_1 - 5\log\lambda_1 - \log\left(1 - e^{-\frac{c_2}{\lambda_1\theta_1}}\right)$$

Writing the similar equation for J_2, λ_2, θ_2, and subtracting, we obtain

$$c_2 = \frac{\lambda_1\lambda_2\theta_1\theta_2}{\lambda_2\theta_2 - \lambda_1\theta_1}\left[\log\frac{J_2}{J_1} + 5\log\frac{\lambda_2}{\lambda_1} + \log\frac{1 - e^{-\frac{c_2}{\lambda_2\theta_2}}}{1 - e^{-\frac{c_2}{\lambda_1\theta_1}}}\right] \qquad (4)$$

This equation contains c_2 in the right-hand member, but in a term which is always relatively small in practice. Since c_2 is a physical constant, a closely approximate value is known, which may be used in this term. A second approximation would rarely modify the result. This satisfactory solution may be simplified by introducing the relation

$$\log\left(1 - e^{-\frac{c_2}{\lambda\theta}}\right) = -e^{-\frac{c_2}{\lambda\theta}} - \frac{1}{2}e^{-\frac{2c_2}{\lambda\theta}} - \frac{1}{3}e^{-\frac{3c_2}{\lambda\theta}} - \ldots\ldots$$

When $\lambda_1\theta_1$ is not too close to $\lambda_2\theta_2$, the first term is sufficient, so that

$$c_2 = \frac{\lambda_1\lambda_2\theta_1\theta_2}{\lambda_2\theta_2 - \lambda_1\theta_1}\left[\log\frac{J_2}{J_1} + 5\log\frac{\lambda_2}{\lambda_1} - e^{-\frac{c_2}{\lambda_2\theta_2}}\right] \qquad (5)$$

This equation is accurate enough for most purposes. It furnishes a very simple means of calculating c_2 from any two observations, there being no restriction to constant temperature or to constant wave length.

The possibility of obtaining a solution for c_2 in a form such as (5) has been indicated by C. N. Haskins.[2] He showed that c_2 could be expressed implicitly in terms of two observations of J, λ, θ, by means of the equation

$$x^n - Cx + C - 1 = 0 \qquad (6)$$

where $x = e^{\frac{c_2}{\lambda_2\theta_2}}$ and $n = \frac{\lambda_2\theta_2}{\lambda_1\theta_1}$. Since n is not an integer, the equation is not solvable algebraically. Some tedious process such as successive approximations is necessary for a solution, so that the method is not as direct, convenient, or rapid as equation (5). Haskins pointed out that while previous methods for calculating c_2 depended upon the use of a plotted curve, the purely mathematical solution depends directly on the experimental data. All graphical errors are thus eliminated. While the graphical errors are smaller than the experimental errors at the present time, it is of interest to know that the former can be avoided.

Equation (5) is particularly interesting because all of the methods which have been used for the determination of c_2 may be derived

[2] Physical Review, 3, p. 476; 1914.

from it very simply. In the first place, if the two observations are at constant wave length, $\lambda_1 = \lambda_2 = \lambda$, equation (5) gives at once:

$$c_2 = \frac{\lambda\theta_1\theta_2}{\theta_2 - \theta_1}\left[\log\frac{J_2}{J_1} - e^{-\frac{c_2}{\lambda\theta_2}}\right] \tag{7}$$

This is the equation for obtaining c_2 from an "isochromatic" curve; that is, observations at a fixed wave length. Isochromatics are used very widely in temperature measurement. In the second place, if temperature is kept constant instead of wave length, $\theta_1 = \theta_2 = \theta$, and equation (3) follows. Equation (3) represents the method which is discussed in detail in this paper. Thirdly, by placing $J_2 = J_1$ in equation (3), the method of equal ordinates appears. Fourthly, by omitting also the exponential correction term, Paschen's equation follows, viz,

$$c_2 = \frac{5\lambda_1\lambda_2\theta}{\lambda_2 - \lambda_1}\log\frac{\lambda_2}{\lambda_1} \tag{8}$$

Fifthly, an equation for the method used at the Reichsanstalt,[3] of observing the maximum J and the value of J at one other λ, follows from equation (3) by writing $\lambda_1 = \lambda$, $J_1 = J$, $\lambda_2 = \lambda_m$, $J_2 = J_m$,

$$4.965\frac{\lambda_m}{\lambda} - (4.965 - e^{-4.965}) = 5\log\frac{\lambda_m}{\lambda} + \log\frac{J_m}{J} \tag{9}$$

Having obtained λ_m, c_2 is given by the relation

$$c_2 = 4.965\,\lambda_m\theta \tag{10}$$

Equation (9) is transcendental, and in practice is solved by preliminary calculation of a table of J_m/J as a function of λ_m/λ. This method is inconvenient physically as well as mathematically. It requires observation of J at one particular point, the maximum of the curve; consequently, the method is impossible at all temperatures for which there is energy absorption at the maximum of the curve.

The chief interest attaching to equation (5) is its generality in relation to other equations, as just discussed. In determining c_2 from two observations, there is not necessarily any advantage in having both temperatures and both wave lengths different. In fact, the requirements of temperature measurement usually necessitate holding the temperature constant at least as long as the time required to observe two values of J. When the temperature is

[3] Warburg, Leithäuser, Hupka and Müller, Ann. 345, p. 625; 1913.

kept constant, and J observed as a function of λ, a so-called "isothermal" is obtained.

In calculating c_2 from isothermals by the method of equal ordinates, it is necessary to draw a curve of J as a function of λ. One of the advantages of the mode of calculation of equation (3) is that no curve is required, and thus all the labor and the errors of plotting and reading a curve are eliminated. Since this method determines c_2 from any two points whatever on the Planck curve, it is applicable when only limited portions of the curve can be observed. This is an important consideration in practice, since absorption bands and other sources of error affect some parts of the curves, and it is sometimes impossible to find a usable ordinate on one side of the maximum equal to a certain ordinate on the other side. These difficulties have limited the general application of the previously used methods, but the two-point method is not affected by them.

Since the two-point method utilizes the observations directly, any abnormal point is very sharply shown up. If a point is in error, for instance, because of errors of observation or errors in the spectrometer setting, this is made evident by an abnormal value of c_2 resulting from the calculation. Each point stands by itself, whereas if a curve is plotted each point is necessarily affected by neighboring points. In the direct two-point calculation only the points affected by the experimental conditions will give wrong values of c_2. A point known to be normal can be used and various others combined with it, then any abnormal values are known to be due to the other points. The constancy of values of c_2 thus calculated from any desired pairs of points is a good criterion to show whether the observations follow Planck's law. Furthermore, variations of c_2 when thus calculated actually give an easy way of investigating the shape of the observed curve as compared with the Planck curve, without plotting either curve. Labor is thus saved, as it is desired to calculate the value of c_2 anyway for its own sake. Considerable information regarding the shape of the curve is given when the two points are both taken on the same side of the maximum, as shown in the example discussed below.

II. APPLICATION OF THE METHOD

The method was tried out on experimental data obtained by Dr. Coblentz, which he very kindly placed at the disposal of the writer. The data were from reliable observations, and the curve which was plotted from them gave consistent values of c_2 by the

method of equal ordinates, with a mean very close to 14 360, practically identical with the general mean for all the curves obtained by Dr. Coblentz.[4] Certain of the calculations by the two-point method give values surprisingly different, as shown below. For convenience in calculation, equation (3) is used in the form

$$c_2 = \frac{2.3026\,\lambda_1\lambda_2\theta}{\lambda_2-\lambda_1}\left[\log_{10}\frac{J_2}{J_1}+5\log_{10}\frac{\lambda_2}{\lambda_1}-0.4343\,e^{-\frac{c_2}{\lambda_2\theta}}\right] \qquad (11)$$

When both points are at the right of the maximum, the exact formula

$$c_2 = \frac{2.3026\,\lambda_1\lambda_2\theta}{\lambda_2-\lambda_1}\left[\log_{10}\frac{J_2}{J_1}+5\log_{10}\frac{\lambda_2}{\lambda_1}+\log_{10}\frac{1-e^{-\frac{c_2}{\lambda_2\theta}}}{1-e^{-\frac{c_2}{\lambda_1\theta}}}\right] \qquad (12)$$

is used. When one of the points is at the left of the maximum, equation (11) is amply accurate.

In the exponential term in the equations, c_2 was taken equal to 14 350. The data were for a temperature of 1350°K, the theoretical curve for which is given in Fig. 1. Values of c_2 calculated by the method of equal ordinates from the observed curve agreed within 1 per cent, with a mean of 14 360. This agreement seemed a satisfactory indication that the data fit the Planck equation within the errors of experiment. The values were very closely checked by calculations from *nearly equal* ordinates, applying equation (11) to the actual observations. The first advantage of the two-point method appears here, viz, the same values of c_2 are obtained by direct calculation from nearly equal ordinates as by the method of equal ordinates, and the graph is dispensed with entirely.

When values of c_2 were calculated from a point of large ordinate on one side of the maximum and a point of small ordinate on the other side, as from points 3 and 5, or 4 and 6 (Fig. 1), there was a fair but not quite so good an agreement with the method of equal ordinates. Thus, the mean of seven such pairs of points gave 14 460.

When calculations were made from two points, both on the same side of the maximum, small departures from the theoretical curve had a magnified effect and very different values were found. Calculations made from nine pairs of points, such as 3 and 4, or 5 and 6 (Fig. 1), gave values of c_2 ranging from 14 530 to 14 930, with a mean of 14 750. These high values were found because

[4] This Bulletin, 13, Scientific Paper No. 284; 1916.

the observed points were slightly below the theoretical curve in two regions, as discussed below.

For calculations from two points at the right of the maximum, equation (12) was used. Some of the calculations were made rigorously, using five significant figures in the intermediate steps. It was found, however, that the use of a slide rule gave the values of c_2 within 1 in the fourth figure; that is, well within the accuracy of observation. It may, therefore, be stated that these calculations can be made very rapidly, obtaining the logarithms and exponentials from tables and performing the multiplications by a slide rule. In calculating the values of c_2, as already stated, a preliminary value of $c_2 = 14\,350$ was used in the term $e^{-\frac{c_2}{\lambda_2\theta}}$. A second approximation would make the values still greater. In the largest values, however, the increase would be only 5 in the fourth significant figure, and as this is within the experimental error the second approximation was not made.

It has been suggested to the author by C. E. Van Orstrand that the reason why the two-point method and the method of equal ordinates give different values is because they weight the observations differently; and that the only comprehensive solution would be by least squares, which would give every observation the proper weight. All of the methods which have been used for calculating c_2 are open to objection on this score. The two-point method, however, is the most powerful in indicating the shape of the curve. It also helps to locate any constant errors, and in this respect seems superior to the method of least squares, which assumes all departures from the theoretical to be accidental. This appears to be an important consideration at the present time, when there are so many variables affecting the observations. The method renders this very service in the case of the observations here discussed.

The excessively high values of c_2 obtained from points entirely on the left or entirely on the right of the maximum indicate that the observed points toward the left of each branch of the curve lie below the theoretical Planck curve. In other words, the curve is slightly sheared, but in such a way that its distortion does not appreciably affect the values calculated by the method of equal ordinates. As a result of this shearing, all the values of c_2 calculated from points on the same side of the maximum are either equal to or higher than the mean found by the method of equal ordinates, although the individual observations do not depart

from the theoretical curve any more than can be accounted for by the large and unavoidable errors of observation. Thus a characteristic of the curve is disclosed by this method of calculation which would be overlooked when the calculations were made by equal ordinates.

It is not known whether this shearing of the curve is wholly a matter of experimental error or a true departure from Planck's equation. A similar characteristic of such curves was found at the Reichsanstalt;[5] that is, different values of c_2 were found from the left and the right sides of the curve (calculated as described above, from one point and J at the maximum). This behavior was found in curves obtained by using a fluorite prism. In the more recent work of Reichsanstalt, very consistent values of c_2 have been obtained, but a quartz prism is used and the observations are all on the left of the maximum, because quartz shows strong absorption at wave lengths above the maximum. It is impossible to say at the present time whether the abnormal values of c_2 obtained from the right side of the curve are due to the inadequacy of the theoretical formula or to the errors of observation.

It is worthy of note that the departures of the above observations from the theoretical curve come in that part of the curve where the Planck and the Wien equations give appreciably different values. In this region the observations fit the Planck law better than the Wien. In the regions of lower wave length and temperature where the two laws agree, they are well known to be very reliable. This is of the greatest importance in various methods of optical pyrometry, which are based on the Wien equation. The recent Reichsanstalt researches do not establish the validity of the Planck modification of the Wien equation, because that work was done in the ranges of wave length and temperature to which both equations apply. Coblentz has already pointed out[6] that the Planck equation has not received a thorough experimental verification. It may be seen from the calculations above that the two-point method shows the characteristics of experimental curves rather more vividly than any previously used criterion.

The Planck radiation equation has considerable importance in various theoretical fields, as well as in its direct experimental application. It is probably as valid to obtain the value of the Planck constant c_2 from its theoretical relations to other constants

[5] ZS Instrumentenkunde, 30, p. 118; 1910.

[6] This Bulletin, 10, p. 3; 1913.

as from radiation data. Prof. Millikan, therefore, seems well justified in his suggestion [7] that a more precise value of c_2 is obtainable on the basis of photo-electric experiments than from radiation data, since higher precision is attained in the former experiments. Of course, the question of which data give the more accurate value could only be answered by an estimate of the constant errors in the two kinds of experiments.

III. DETERMINATION OF SPECIAL POINTS ON THE PLANCK CURVE

For certain points on a Planck isothermal curve, Wien's displacement law holds. The law is expressed by the equation,

$$c_2 = K\lambda_s\theta \qquad (13)$$

where K is a constant and λ_s is the λ coordinate of the point. Such special points are the maximum of the curve, for which $K = 4.965$; the center of gravity, for which $K = 2.701$; and the points of inflection. By the use of the displacement law, c_2 could theoretically be obtained from an observation merely of the λ for one of these special points. As a matter of fact, however, none of these points can be taken directly from the curve with accuracy, so no additional ways of obtaining c_2 are really provided in this way. The earliest observers, such as Lummer and Pringsheim, did calculate c_2 from observations of λ at the maximum, but the method would no longer be considered. P. D. Foote has proposed [8] using the center of gravity similarly, obtaining the λ coordinate by mechanical integration of the curve. Such a method would involve all the errors of the curve and of the integration, the former being particularly formidable since a part of the curve known to be affected by absorption bands could not be omitted. Since only a finite portion of the curve can be integrated mechanically, a correction has to be computed by successive approximations. The method is laborious and not highly accurate, and is, in fact, not recommended by its author for actual use.

Not only are these special points unsuitable as means of determining c_2, but they may themselves be obtained most accurately and conveniently by the very process which is recommended in this paper for obtaining c_2. Any of the points for which the displacement law holds may be calculated from any two observed

[7] Phys. Rev., 7, p. 378; 1916.

[8] This Bulletin, 12, p. 479; 1915.

points. Substituting from equation (13) in equation (3), and letting $K=\frac{1}{K'}$,

$$\lambda_s=\frac{K'\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left[\log\frac{J_2}{J_1}+5\log\frac{\lambda_2}{\lambda_1}-e^{-\frac{c_2}{\lambda_2\theta}}\right]. \qquad (14)$$

This is the same as equation (3), with K' substituted for θ. For the maximum of the curve, $K'=0.2014$; for the center of gravity, $K'=0.3702$. This means that the maximum of the curve need not be observed in order to determine it, and, in fact, it could not be determined with nearly as great accuracy by direct observation as by calculation from several pairs of reliable points. Also, the whole curve need not be observed in order to determine the center of gravity. Practically a single equation suffices to determine c_2 and all the special points from any two observed points.

It may be noted that only a very rough value of the temperature need be known in calculating the special points by equation (14). One additional possibility of the two-point method is, of course, the determination of temperature. Solving equation (3) for θ instead of for c_2,

$$\theta=\frac{c_2(\lambda_2-\lambda_1)}{\lambda_1\lambda_2\left[\log\frac{J_2}{J_1}+5\log\frac{\lambda_2}{\lambda_1}-e^{\frac{-c_2}{\lambda_2\theta}}\right]} \qquad (15)$$

This requires knowledge of the value of c_2 and a roughly approximate value of θ. This would be a more accurate method of temperature measurement than the use of the displacement law for the maximum of the curve. It might be a useful method in certain experiments where the proper apparatus was set up.

IV. SUMMARY

The constant c_2 of Planck's radiation equation has heretofore been obtained from radiation data by processes involving the use of a graph. It may be determined very simply and directly from two observations at any wave lengths and temperatures. The formula for the case of constant temperature is only slightly different from the familiar equation for equal ordinates. The method eliminates all graphical difficulties, and is less limited by experimental conditions such as absorption bands in the air and the prism.

A mathematical discussion has been given, showing how very simply the equations for all the known methods of calculating c_2 may be deduced from a general solution of Planck's equation.

On applying the general method of calculating in terms of two points to an experimental energy curve, the power of the method in investigating the shape of the curve is strikingly shown. The departures of the observations from the Planck law are more vividly indicated than in other methods of calculation. The departures of individual points and of limited portions of the curve from the theoretical values stand out by themselves.

Points on the Planck curve for which Wien's displacement law holds, in particular the maximum of the curve, have been considered as furnishing additional ways of determining c_2. Such methods are debarred by lack of accuracy and, in fact, these special points may themselves be obtained most accurately and conveniently by the very process which is recommended in this paper for obtaining c_2. Substantially the same simple equation suffices to determine c_2 and all the special points.

WASHINGTON, May 10, 1916.